

The University of Chicago

**DIVISION ALGEBRAS ASSOCIATED WITH
AN EQUATION WHOSE GROUP HAS
FOUR GENERATORS**

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

MINA SPIEGEL REES

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS

Reprinted from
AMERICAN JOURNAL OF MATHEMATICS
Vol. LIV, No. 1, 1932

The University of Chicago

DIVISION ALGEBRAS ASSOCIATED WITH
AN EQUATION WHOSE GROUP HAS
FOUR GENERATORS

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS.

BY
MINA SPIEGEL REES

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS

Reprinted from
AMERICAN JOURNAL OF MATHEMATICS
Vol. LIV, No. 1, 1932



TABLE OF CONTENTS.

1. INTRODUCTION.
 - a. Historical.
 - b. Statement of problem.
2. ALGEBRA Σ DERIVED FOR AN EQUATION WHOSE GROUP HAS FOUR GENERATORS.
 - a. Relations satisfied by roots of the given equation.
 - b. Multiplication table for Σ .
 - c. Definition of correspondence between elements in Σ .
3. ASSOCIATIVITY CONDITIONS FOR Γ .
 - a. Preliminary definitions and formulas.
 - b. $A'B' = (AB)'$ obtained by means of equations (42)-(47).
 - c. $\gamma = \gamma'$ obtained by means of equation (50).
 - d. $A^{(h)}\gamma = \gamma A$ obtained by means of equations (59), (60), (61).
4. EXAMPLE.
 - a. A group of order 32.
 - b. Associativity conditions.
 - c. Multiplication table for an algebra.



Digitized by the Internet Archive
in 2026

DIVISION ALGEBRAS ASSOCIATED WITH AN EQUATION WHOSE GROUP HAS FOUR GENERATORS.

By MINA S. REES.

1. *Introduction.* The construction of all division algebras is the outstanding problem in the theory of linear algebras. In the history of this problem, the procedure has been to examine first the necessary and sufficient conditions that the constructed algebra be associative; and second, the conditions that it be a division algebra. Since L. E. Dickson's announcement¹ in 1905 of his discovery of a system of non-commutative division algebras, he has published three papers examining associativity conditions for such algebras. In the first of these papers,² algebras associated with cyclic equations were discussed; in the second,³ general results were obtained for the abelian non-cyclic case, and for that type of non-abelian case where the Galois group of the basic equation was a solvable group. This study was carried forward by Williamson,⁴ in a paper in which detailed associativity conditions were established for the general case of a two generator group, and for a significant special case of the three generator group. In a third paper in 1930, Dickson⁵ gave a simplification of his previous method by which he reached specific results for the general case of the two and three generator problems. The second line of investigation, the study of the conditions under which a given associative algebra is a division algebra, has been advanced by Dickson,⁶ Wedderburn,⁷ and Albert.⁸ In addition to these two aspects of the problem, there is a third phase which is concerned with normal division algebras. It is known that any division algebra can be normalized by a suitable

1. *Bulletin of the American Mathematical Society*, Vol. 22 (1905-06), p. 442.

2. "Linear Associative Algebras and Abelian Equations," *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 31-46.

3. "New Division Algebras," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 207-34.

4. "Associativity of Division Algebras," *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 111-25.

5. "Construction of Division Algebras," *Transactions of the American Mathematical Society*, Vol. 32 (1930), pp. 319-34.

6. See notes (2) and (3).

7. "A Type of Primitive Algebra," *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 162-166.

8. Abstracts in the *Bulletin of the American Mathematical Society*, Vol. 36 (1930), p. 45; p. 198; p. 804.

extension of the field, so that its order is the square of an integer,⁹ and a third line of study has established that all division algebras of orders four,¹⁰ nine,¹¹ or sixteen¹² are among those for which associativity conditions have been found.

The present paper continues the investigation of associativity conditions, by considering the algebras related to an equation whose Galois group has four independent generators. Using Dickson's most recent simplification of his previous method,¹³ we set up necessary and sufficient conditions that these algebras be associative. Since the method is inductive, all the results which Dickson obtained for the two or three generator cases¹⁴ will be presupposed. The notation of the present paper is the same as in Dickson's except that $v, y, z, w, \epsilon, \delta$ of his paper are replaced by $z_1, w_1, z_2, w_2, \delta_1, \delta_2$ respectively. Numbered theorems and formulas in square brackets refer to theorems and formulas in his paper.

2. Algebra Σ derived for an equation whose group has four generators.

Let $f(\xi)=0$ be an equation of degree $pqsh$ which is irreducible in F and has the roots

$$(1) \quad \chi^{a_1} \{ \psi^{a_2} [\phi^{a_3} (\theta^{a_4} (i))] \} \quad (a_1 < h, a_2 < s, a_3 < p, a_4 < q),$$

where χ, ψ, ϕ, θ are polynomials in i with coefficients in F , and the superscripts denote iteratives, not exponents.

Throughout this paper, the brackets $\{ \}$ will mean that each of the included symbols is to be interpreted as a polynomial in the succeeding symbols, where the final argument i is suppressed. Thus $\psi\{\theta\phi\}$ means $\psi[\theta(\phi(i))]$.

Let the roots satisfy the following relations:

$$(2) \quad \theta^q = i, \quad \phi^p = \theta^e, \quad \psi^s = \phi^b \{ \theta^a \}, \quad \chi^h = \psi^l \{ \phi^d \theta^c \},$$

$$(3) \quad \theta\{\phi\} = \phi\{\theta^w\}, \quad \theta\{\psi\} = \psi\{\phi^{z_1} \theta^{w_1}\}, \quad \phi\{\psi\} = \psi\{\phi^{z_2} \theta^{w_2}\},$$

$$(4) \quad \theta\{\chi\} = \chi\{\psi^{u_1} \phi^{v_1} \theta^{w_1}\}, \quad \phi\{\chi\} = \chi\{\psi^{u_2} \phi^{v_2} \theta^{w_2}\}, \quad \psi\{\chi\} = \chi\{\psi^{u_3} \phi^{v_3} \theta^{w_3}\},$$

where q, p, s, h are the least positive integers for which relations of the type (2) hold.

9. Dickson, *Algebren und ihre Zahlentheorie*, p. 138.

10. See note (9) above. *Loc. cit.*, p. 46.

11. Wedderburn, "On Division Algebras," *Transactions of the American Mathematical Society*, Vol. 22 (1921), pp. 129-35.

12. Albert, "A Determination of All Division Algebras in Sixteen Units," *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 253-60.

13. See note (5), page 51.

14. *Ibid.*

Let Σ be the algebra over F having the basis $i^m j^n k^t o^f$ ($m \leq pqs - 1$, $n \leq q - 1$, $t \leq p - 1$, $f \leq s - 1$) such that $f(i) = 0$ and

$$(5) \quad j^q = g(i), \quad j^r P(i) = P(\theta^r) j^r;$$

$$(6) \quad k^p = \beta(i) j^e, \quad k^r P(i) = P(\phi^r) k^r, \quad k^r j^t = [j^{(r)}]^t k^r,$$

where $j^{(r)}$ is defined by [16];

$$(7) \quad \begin{aligned} o^s &= \sigma(i) j^a k^b, & o^r P(i) &= P(\psi^r) o^r, \\ o^r j^t &= [j^{(r)}]^t o^r, \\ o^r k^t &= [k^{(r)}]^t o^r, \end{aligned}$$

where

$$J \equiv j' \equiv \delta_1 j^{w_1} k^{z_1}, \quad K \equiv k' \equiv \delta_2 j^{w_2} k^{z_2}$$

and

$$j^{(r)} = \delta_1 (\psi^{r-1}) (J^{w_1} K^{z_1})^{(r-2)}, \quad k^{(r)} = \delta_2 (\psi^{r-1}) (J^{w_2} K^{z_2})^{(r-2)}$$

are found by [64]. We assume g , β , and σ to be different from zero.

Σ may be obtained from [Theorem 5] by replacing F by the field F_1 , derived by adjoining to F the elementary symmetric functions of all of the roots $\psi^{a_2} \{ \phi^{a_3} \theta^{a_4} \}$. Then the latter are roots of an equation irreducible in F_1 of degree pqs .¹⁵ Therefore by [Theorem 5], Σ has the above basis. Assume the conditions required in [Theorem 5] so that Σ be associative. We see that (5), (6), (7) are consequences of the associative law and the following set R of relations:

$$(8) \quad j^q = g(i), \quad ji = \theta(i) j, \quad f(i) = 0,$$

$$(9) \quad k^p = \beta(i) j^e, \quad ki = \phi(i) k, \quad kj = \alpha(i) j^x k,$$

$$(10) \quad o^s = \sigma(i) j^a k^b, \quad oi = \psi(i) o, \quad oj = \delta_1(i) j^{w_1} k^{z_1} o, \quad ok = \delta_2(i) j^{w_2} k^{z_2} o.$$

These relations enable us to reduce any product AB to C of the same form, where A , B , and C are polynomials of Σ . Let R' denote the set of like relations, with i, j, k, o replaced by i', j', k', o' which are defined to be

$$(11) \quad \begin{aligned} I &= i' = \chi(i), & J &= j' = \epsilon_1(i) j^{v_1} k^{v_1} o^{u_1}, \\ K &= k' = \epsilon_2(i) j^{v_2} k^{v_2} o^{u_2}, \\ O &= o' = \epsilon_3(i) j^{v_3} k^{v_3} o^{u_3}. \end{aligned}$$

If relations R' hold, it follows¹⁶ that

$$C' = (AB)' = A'B', \quad (A + B)' = A' + B'.$$

15. See note (3), page 51. *Loc. cit.*, § 6.

16. For further details see Dickson, "Construction of Division Algebras," *Transactions of the American Mathematical Society*, Vol. 32 (1930), p. 322.

Thus [Theorem 1] will enable us to construct an associative algebra Γ provided the correspondence defined in Σ is such that the following conditions are satisfied:

(a) Relations R' hold, so that

$$(AB)^{(r)} = A^{(r)}B^{(r)}, \quad (A+B)^{(r)} = A^{(r)} + B^{(r)}.$$

(b) An element γ is defined for which $\gamma = \gamma'$.

(c) $A^{(h)}\gamma = \gamma A$ for every A in Σ .

3. *Associativity Conditions for Γ .* We investigate first the conditions for R' to be true. It is easily seen that $(8'_2)$, $(9'_2)$ and $(10'_2)$ are satisfied identically by reason of the definition of the correspondence in Σ . Insert the values (11) in $(8'_2)$. Then, by (4_1) ,

$$\begin{aligned} JI &= \epsilon_1 \chi \{ \psi^{u_1} \phi^{v_1} \theta^{u_2} \} j^{y_1} k^{v_1} o^{u_1}, \\ &= \epsilon_1 \theta \{ \chi \} j^{y_1} k^{v_1} o^{u_1} = \theta \{ \chi \} J. \end{aligned}$$

Similarly, by using subscripts 2 and 3, relations $(9'_2)$ and $(10'_2)$ are established by (4_2) and (4_3) respectively.

Preliminary Definitions and Formulas. We shall use the following abbreviations, corresponding to [36], [38], [39] respectively:

$$\begin{aligned} (12) \quad B_{m,n}(i) &\equiv \pi_n(\phi^{m-1}) \pi_{x_n}(\phi^{m-2}) \pi_{x^2_n}(\phi^{m-3}) \cdots \pi_{x^{m-1}_n}(i), \\ \pi_n(i) &= \alpha \alpha(\theta^x) \cdots \alpha(\theta^{(n-1)x}), \quad B_{m,0} = B_{0,n} = 1, \end{aligned}$$

$$(13) \quad h(t) \equiv \sum_{n=0}^{t-1} x^{nz_1} w_1, \quad h(0) = 0,$$

$$(14) \quad C_d(i) \equiv \prod_{n=0}^{d-1} \delta_1 \{ \phi^{nz_1} \theta^{h(n)} \} B_{nz_1, w_1} \{ \theta^{h(n)} \}, \quad C_0(i) \equiv 1,$$

and the formulas [35] and [37]

$$(15) \quad k^m j^n = B_{m,n} j^{x^m n} k^m,$$

$$(16) \quad \delta j^w k^x \epsilon j^y k^v = \delta \epsilon \{ \phi^z \theta^w \} B_{z,y}(\theta^w) j^{w+xz} k^{z+v},$$

where δ and ϵ are polynomials in i . Then from (10_3) we see by induction on n , that

$$(17) \quad o j^n = C_n(i) j^{h(n)} k^{nz_1} o.$$

Similarly, if $H(n)$ is derived from $h(n)$ by using subscripts 2 throughout, and $D_d(i)$ from $C_d(i)$ by using subscripts 2 and substituting $H(n)$ for $h(n)$, we have

$$(18) \quad ok^n = D_n(i) j^{H(n)} k^{nz_2} o.$$

Formula [58] is

$$t(m, n) \equiv h(m) + x^{mz_1} H(n).$$

From (17) and (18), by induction on m , we can deduce formulas corresponding to (15). We find, in fact

$$(20) \quad \begin{cases} o^m j^n = A_{mn}(i) j^{A(m,n)} k^{L(m,n)} o^m, \\ o^m k^n = D_{mn}(i) j^{R(m,n)} k^{S(m,n)} o^m, \end{cases}$$

where (21)-(24) hold. The recursion formulas

$$(21) \quad \begin{cases} A(1, n) = h(n), & A(m+1, n) = t[A(m, n), L(m, n)], \\ L(1, n) = nz_1, & L(m+1, n) = A(m, n)z_1 + L(m, n)z_2, \end{cases}$$

$$(22) \quad \begin{cases} R(1, n) = H(n), & R(m+1, n) = t[R(m, n), S(m, n)], \\ S(1, n) = nz_2, & S(m+1, n) = R(m, n)z_1 + S(m, n)z_2, \end{cases}$$

with

$$\begin{aligned} A(0, n) &= S(0, n) = n, & L(0, n) &= R(0, n) = 0, \\ A(m, 0) &= L(m, 0) = R(m, 0) = S(m, 0) = 0; \end{aligned}$$

and the abbreviations

$$(23) \quad F_{m,n}(i) = C_m D_n \{ \phi^{mz_1} \theta^{h(m)} \} B_{mz_1, H(n)} \{ \theta^{h(m)} \},$$

$$(24) \quad \begin{cases} A_{mn}(i) = \prod_{r=0}^{m-1} F_{A(r,n), L(r,n)} \{ \psi^{m-1-r} \}, \\ D_{mn}(i) = \prod_{r=0}^{m-1} F_{R(r,n), S(r,n)} \{ \psi^{m-1-r} \}, \end{cases}$$

with

$$A_{1n}(i) = C_n(i), \quad D_{1n}(i) = D_n(i), \quad A_{0n}(i) \equiv D_{0n}(i) \equiv 1,$$

actually define the functions that occur in (20).

Applying (7₂), (20), (16), we have

$$\begin{aligned} KJ &= \epsilon_2 j^{u_2} k^{v_2} o^{u_2} \epsilon_1 j^{v_1} k^{v_1} o^{u_1}, \\ &= \epsilon_2 \epsilon_1 \{ \psi^{u_2} \phi^{v_2} \theta^{u_2} \} A_{u_2 v_1} \{ \phi^{v_2} \theta^{u_2} \} D_{u_2 v_1} \{ \phi^{v_2+L(u_2, y_1)} \theta^{u_2+x^{v_2} A(u_2, y_1)} \} \\ (25_1) \quad &\times B_{v_2, A(u_2, y_1)} \{ \theta^{v_2} \} B_{v_2+L(u_2, y_1), R(u_2, v_1)} \{ \theta^{u_2+x^{v_2} A(u_2, y_1)} \} \\ &\times j^{u_2+x^{v_2} A(u_2, y_1)+x^{v_2+L(u_2, y_1)} R(u_2, v_1)} k^{v_2+L(u_2, y_1)+S(u_2, v_1)} o^{u_2+u_1}. \end{aligned}$$

Define

$$\begin{aligned} \mathcal{K}_{ab}(r, t) &\equiv v_a + \sum_{n=1}^{r-1} L(nu_a, y_b) + \sum_{m=1}^{t-1} S(mu_a, v_b), & \mathcal{K}_{ab}(0, 0) &= 0, \\ (26) \quad \mathcal{L}_{ab}(r, t) &\equiv y_a + \sum_{n=1}^{r-1} x^{\mathcal{K}_{ab}(n, n)} A(nu_a, y_b) \\ &\quad + \sum_{m=1}^{t-1} x^{\mathcal{K}_{ab}(m+1, m)} R(mu_a, v_b), & \mathcal{L}_{ab}(0, 0) &= 0, \end{aligned}$$

where summations from one to zero are equal to zero;

$$(27) \quad \mathcal{K}_{ab}(2, 2) \equiv \mathcal{F}_{ab}, \quad \mathcal{Q}_{ab}(2, 2) \equiv ab,$$

$$(28) \quad \mathcal{H}_{ab}(i) \equiv \epsilon_a \epsilon_b \{ \psi^{u_a} \phi^{v_a} \theta^{y_a} \} \mathcal{A}_{u_a v_b} \{ \phi^{v_a} \theta^{y_a} \} \\ \times \mathcal{D}_{u_a v_b} \{ \phi \mathcal{K}^{u_b(2,1)} \theta^{2,1} \} B_{v_a, A(u_a, v_b)} \{ \theta^{y_a} \} \\ \times B_{\mathcal{K}_{ab}(2,1), R(u_a, v_b)} \{ \theta^{2,1} \};$$

$$(29) \quad \mathcal{G}^{(d)}(i) \equiv \prod_{n=0}^{d-1} \epsilon \{ \psi^{n u} \phi \mathcal{K}^{(n,n)} \theta^{2(n,n)} \} \mathcal{A}_{n u, y} \{ \phi \mathcal{K}^{(n,n)} \theta^{2(n,n)} \} \\ \times B_{\mathcal{K}^{(n,n)}, A(n u, y)} \{ \theta^{2(n,n)} \} B_{\mathcal{K}^{(n+1,n)}, R(n u, v)} \{ \theta^{2(n+1,n)} \} \\ \times \mathcal{D}_{n u, v} \{ \phi \mathcal{K}^{(n+1,n)} \theta^{2(n+1,n)} \} \quad (d > 0), \\ \mathcal{G}^{(0)}(i) \equiv 1, \quad \mathcal{G}^{(1)}(i) \equiv \epsilon,$$

where a subscript on the $\mathcal{G}$ is to be interpreted as the subscript of every ϵ , u , v , y , and the double subscript of every $\mathcal{K}$, $\mathcal{Q}$. Using these abbreviations we obtain a generalized statement of (25₁)

$$(25_2) \quad \epsilon_a j^{y_a} k^{v_a} \epsilon_b j^{y_b} k^{v_b} \theta^{u_b} = \mathcal{H}_{ab}(i) j^{\mathcal{E}_{ab}} k^{\mathcal{F}_{ab}} \theta^{u_a + u_b}.$$

We have immediately

$$(30) \quad \begin{aligned} KJ &= \mathcal{H}_{21}(i) j^{\mathcal{E}_{21}} k^{\mathcal{F}_{21}} \theta^{u_2 + u_1}, \\ OJ &= \mathcal{H}_{31}(i) j^{\mathcal{E}_{31}} k^{\mathcal{F}_{31}} \theta^{u_3 + u_1}, \\ OK &= \mathcal{H}_{32}(i) j^{\mathcal{E}_{32}} k^{\mathcal{F}_{32}} \theta^{u_3 + u_2}. \end{aligned}$$

By induction on d , we obtain from (11₂) and (25₁)

$$(31) \quad J^d = \mathcal{G}_1^{(d)} j^{\mathcal{Q}_{11}(d,d)} k^{\mathcal{K}_{11}(d,d)} \theta^{u_1 d}.$$

K^d and O^d are obtained from J^d by using subscripts 2 and 3 respectively.

In the discussion which follows, the numbers in the left-hand column of the table given below will be used as superscripts on $\mathcal{H}$, $\mathcal{E}$, $\mathcal{F}$, to indicate the results obtained from formulas (27) and (28) by making the tabulated substitutions. In (7) and (8), subscripts (n, t) and (m, n, t) respectively, will be used on $\mathcal{H}$, $\mathcal{E}$, $\mathcal{F}$. The last line of the table gives superscripts (9), (10), (11) when subscripts 1, 2, 3 respectively are used on y , v , and u .

Superscript	ϵ_a	y_a	v_a	u_a	ϵ_b	y_b	v_b	u_b
(1)	$\mathfrak{g}_1^{(x)}$	$\mathcal{Q}_{11}(x, x)$	$\mathcal{K}_{11}(x, x)$	$u_1 x$	ϵ_2	y_2	v_2	u_2
(2)	$\mathfrak{g}_2^{(z_1)}$	$\mathcal{Q}_{22}(z_1, z_1)$	$\mathcal{K}_{22}(z_1, z_1)$	$u_2 z_1$	ϵ_3	y_3	v_3	u_3
(3)	$\mathfrak{g}_1^{(w_1)}$	$\mathcal{Q}_{11}(w_1, w_1)$	$\mathcal{K}_{11}(w_1, w_1)$	$u_1 w_1$	$\mathcal{G}^{(2)}$	$\mathcal{E}^{(2)}$	$\mathcal{F}^{(2)}$	$u_2 z_1 + u_3$
(4)	$\mathfrak{g}_2^{(z_2)}$	$\mathcal{Q}_{22}(z_2, z_2)$	$\mathcal{K}_{22}(z_2, z_2)$	$u_2 z_2$	ϵ_3	y_3	v_3	u_3
(5)	$\mathfrak{g}_1^{(w_2)}$	$\mathcal{Q}_{11}(w_2, w_2)$	$\mathcal{K}_{11}(w_2, w_2)$	$u_1 w_2$	$\mathcal{G}^{(4)}$	$\mathcal{E}^{(4)}$	$\mathcal{F}^{(4)}$	$u_2 z_2 + u_3$
(6)	$\mathfrak{g}_1^{(a)}$	$\mathcal{Q}_{11}(a, a)$	$\mathcal{K}_{11}(a, a)$	$u_1 a$	$\mathfrak{g}_2^{(b)}$	$\mathcal{Q}_{22}(b, b)$	$\mathcal{K}_{22}(b, b)$	$u_2 b$
(7)	$\mathfrak{g}_2^{(n)}$	$\mathcal{Q}_{22}(n, n)$	$\mathcal{K}_{22}(n, n)$	$u_2 n$	$\mathfrak{g}_3^{(t)}$	$\mathcal{Q}_{33}(t, t)$	$\mathcal{K}_{33}(t, t)$	$u_3 t$
Subscript (n, t)	$\mathfrak{g}_1^{(m)}$	$\mathcal{Q}_{11}(m, m)$	$\mathcal{K}_{11}(m, m)$	$u_1 m$	$\mathcal{G}^{(7)}_{(n, t)}$	$\mathcal{E}^{(7)}_{(n, t)}$	$\mathcal{F}^{(7)}_{(n, t)}$	$u_2 n + u_3 t$
Subscript (m, n, t)	$\Omega_{y, n, u, n-2}$	η	λ	τ	ρ	c	d	l
(9), (10), or (11)								

where the arguments of η, λ, τ , in (9), (10), (11) are $(y, v, u, h-2)$. We shall need the formula

$$(33) \quad k^{np} = \beta_n(i) j^{en}, \quad \beta_n(i) = \beta \beta(\theta^e) \cdot \dots \beta(\theta^{e(n-1)}),$$

which is [42] and the formula

$$(34) \quad o^{ns} = \sigma_n(i) j^{r(n)} k^{bn},$$

where

$$r(n) = \sum_{t=0}^{n-1} x^{tb} a, \quad r(0) = 0,$$

$$\sigma_n(i) = \prod_{t=0}^{n-1} \sigma \{ \phi^{tb} \theta^{r(t)} \} B_{tb,a} \{ \theta^{r(t)} \}, \quad \sigma_0(i) \equiv 1.$$

which is obtained by induction on s , using (7₁) and (16).

In finding the associativity conditions for our algebra, we must discuss ten conditions of the type

$$(35) \quad P(i) j^U k^V o^W = S(i) j^X k^Y o^Z.$$

Since s is the least power of o which gives a polynomial in i, j, k , we have

$$(36) \quad Z - W = c_1 s, \quad c_1 \text{ an integer.}$$

Replacing $o^{c_1 s}$ by its value (34), and applying (16), we have

$$P(i) j^U k^V = S(i) j^X k^Y \sigma_{c_1}(i) j^{r(c_1)} k^{bc_1},$$

$$= S(i) \sigma_{c_1} \{ \phi^Y \theta^X \} B_{Y, r(c_1)} \{ \theta^X \} j^{X+Yr(c_1)} k^{Y+bc_1}.$$

By comparing exponents of k , since p is the least power of k which gives a polynomial in i and j , we see that

$$(37) \quad Y + bc_1 - V = c_2 p, \quad c_2 \text{ an integer.}$$

Replacing $k^{c_2 p}$ by its value (33), applying (5₂), and comparing exponents of j , we have

$$(38) \quad X + x^Y r(c_1) + ec_2 - U = c_3 q, \quad c_3 \text{ an integer.}$$

Replacing j^{eq} by its value from (5₁), we see that (35) is equivalent to the associativity condition

$$(39) \quad P(i) = S(i) \sigma_{c_1} \{ \phi^Y \theta^X \} B_{Y, r(c_1)} \{ \theta^X \} \beta_{c_2} \{ \theta^{X+Yr(c_1)} \} g^{c_3}.$$

In applying these results to special cases, identify the polynomials $P(i)$ and $S(i)$ with the given coefficients, and the exponents U, V, W, X, Y, Z , with

the given exponents in such a way that the c_1 determined by (36) is positive. Then it is formally possible that c_2 determined by (37) is negative. In this case we would have

$$V - Y - bc_1 = -c_2p.$$

Proceeding as above, replacing k^{-c_2p} by its value (33) and comparing exponents of k , we find

$$X + x^Y r(c_1) - U + ec_2 = c_3q$$

which agrees with (38). The associativity condition for this case is

$$(40) \quad P(i)\beta_{-c_2}\{\theta^U\} = S(i)\sigma_{c_1}\{\phi^Y\theta^X\}B_{Y,r(c_1)}\{\theta^X\}g^{c_3}.$$

We observe, therefore, that if we define

$$(41) \quad \beta_{-n} \equiv [\beta_n\{\theta^{-en}\}]^{-1}$$

relation (40) follows from (39) since

$$X + x^Y r(c_1) + ec_2 \equiv U \pmod{q}, \quad \theta^q = i.$$

We may now apply (36), (37), (38), (39), (41) as formulas to determine the desired associativity conditions. The relations corresponding to (35) will be stated, and the evaluation of the formulas left for special cases. Thus the associativity conditions for our algebra will be derivable from relations (42), (43), (44), (45), (46), (47), (50), (59), (60), (61) given below, by the application of formulas (39), (41).

We will discuss first $(9_3')$, $(10_3')$, $(10_4')$, $(8_1')$, $(9_1')$, $(10_1')$.

Condition that $KJ = \alpha(\chi)J^wK$. By (30₁), (31) and (32), this reduces to

$$\mathcal{A}_{21}(i)j^{\mathcal{E}_{21}k}\mathcal{F}_{21O}u_2+u_1 = \alpha(\chi)\mathcal{G}_1^{(w)}(i)j^{\mathcal{Q}_{11}(w,w)}k^{\mathcal{K}_{11}(w,w)}O^{u_1w}\epsilon_2(i)j^{y_2}k^{v_2O}u_2.$$

Hence

$$(42) \quad \mathcal{A}_{21}(i)j^{\mathcal{E}_{21}k}\mathcal{F}_{21O}u_2+u_1 = \alpha(\chi)\mathcal{A}^{(1)}(i)j^{\mathcal{E}^{(1)}k}\mathcal{F}^{(1)}O^{u_1w+u_2}.$$

Condition that $OJ = \epsilon_1(\chi)J^{w_1}K^{z_1}O$ (subscripts 1 on w and z).

$$\begin{aligned} \mathcal{A}_{31}(i)j^{\mathcal{E}_{31}k}\mathcal{F}_{31O}u_3+u_1 &= \epsilon_1(\chi)\mathcal{G}_1^{(w)}j^{\mathcal{Q}_{11}(w,w)}k^{\mathcal{K}_{11}(w,w)}O^{u_1w} \\ &\quad \times \mathcal{G}_2^{(z)}j^{\mathcal{Q}_{22}(z,z)}k^{\mathcal{K}_{22}(z,z)}O^{u_2z}\epsilon_3j^{y_3}k^{v_3O}u_3. \end{aligned}$$

Hence

$$(43) \quad \mathcal{A}_{31}(i)j^{\mathcal{E}_{31}k}\mathcal{F}_{31O}u_3+u_1 = \epsilon_1(\chi)\mathcal{A}^{(3)}j^{\mathcal{E}^{(3)}k}\mathcal{F}^{(3)}O^{u_1w_1+u_2z_1+u_3}.$$

Condition that $OK = \epsilon_2(\chi)J^{w_2}K^{z_2}O$ (subscripts 2 on w and z).

$$(44) \quad \mathcal{H}_{32}(i) j^{\mathcal{E}_{32}} k^{\mathcal{F}_{32}} o^{u_3+u_2} = \epsilon_2(\chi) \mathcal{H}^{(5)} j^{\mathcal{E}^{(5)}} k^{\mathcal{F}^{(5)}} o^{u_1 u_2 + u_2 u_3 + u_3}.$$

Condition that $J^q = g(\chi)$.

$$(45) \quad \mathcal{G}_1^{(q)} j^{\mathcal{Q}_{11}(q,q)} k^{\mathcal{K}_{11}(q,q)} o^{u_1 q} = g(\chi).$$

Condition that $K^p = \beta(\chi) J^c$.

$$(46) \quad \mathcal{G}_2^{(p)} j^{\mathcal{Q}_{22}(p,p)} k^{\mathcal{K}_{22}(p,p)} o^{u_2 p} = \beta(\chi) \mathcal{G}_1^{(e)} j^{\mathcal{Q}_{11}(e,e)} k^{\mathcal{K}_{11}(e,e)} o^{u_1 e}.$$

Condition that $O^s = \sigma(\chi) J^a K^b$.

$$\mathcal{G}_3^{(s)} j^{\mathcal{Q}_{33}(s,s)} k^{\mathcal{K}_{33}(s,s)} o^{u_3 s} = \sigma(\chi) \mathcal{G}_1^{(a)} j^{\mathcal{Q}_{11}(a,a)} k^{\mathcal{K}_{11}(a,a)} o^{u_1 a} \\ \mathcal{G}_2^{(b)} j^{\mathcal{Q}_{22}(b,b)} k^{\mathcal{K}_{22}(b,b)} o^{u_2 b}.$$

Hence

$$(47) \quad \mathcal{G}_3^{(s)} j^{\mathcal{Q}_{33}(s,s)} k^{\mathcal{K}_{33}(s,s)} o^{u_3 s} = \sigma(\chi) \mathcal{H}^{(6)} j^{\mathcal{E}^{(6)}} k^{\mathcal{F}^{(6)}} o^{u_1 a + u_2 b}.$$

We have now given all the conditions derived from (8'), (9'), (10'), in the form (35). Choose as γ ,

$$(48) \quad \gamma = \rho(i) j^c k^d o^l, \quad [\rho(i) \neq o].$$

Using (25), (31), (32), we establish the formula

$$(49) \quad J^m K^n O^t = \mathcal{H}_{(m,n,t)}^{(8)} j^{\mathcal{E}_{(m,n,t)}^{(8)}} k^{\mathcal{F}_{(m,n,t)}^{(8)}} o^{u_1 m + u_2 n + u_3 t}.$$

Condition that $\gamma = \gamma'$ [subscripts (c, d, l) on $\mathcal{H}, \mathcal{E}, \mathcal{F}$].

$$\rho(i) j^c k^d o^l = \rho(\chi) J^c K^d O^l,$$

$$(50) \quad \rho(i) j^c k^d o^l = \rho(\chi) \mathcal{H}^{(8)} j^{\mathcal{E}^{(8)}} k^{\mathcal{F}^{(8)}} o^{u_1 c + u_2 d + u_3 l}.$$

In order to construct our associative algebra Γ , we need the further condition that

$$(51) \quad A^{(h)} \gamma = \gamma A$$

for all A in Σ . Evidently this condition holds for $A = P(i)$, by reason of our definition of γ . For, by (5), (6), (7), and (24),

$$\gamma P(i) = \rho P\{\psi^l \phi^d \theta^c\} j^c k^d o^l, \\ = P(\chi^h) \gamma = P^{(h)} \gamma.$$

We must now find the condition that (51) holds when A takes the values j, k, o , respectively.

Condition that $j^{(h)} \gamma = \gamma j$ (subscripts 1 on every ϵ, u, v, y).

$$j' = \epsilon j^u k^v o^u, \quad j'' = \epsilon(\chi) J^u K^v O^u.$$

We need a formula of the type

$$(52) \quad (J^m K^n O^t)^{(r)} = \Omega_{(m,n,t,r)}(i) j^{\eta(m,n,t,r)} k^{\lambda(m,n,t,r)} o^{\tau(m,n,t,r)}.$$

Hence

$$(53) \quad \Omega_{(m,n,t,r)}(\chi) J^{\eta(m,n,t,r)} K^{\lambda(m,n,t,r)} O^{\tau(m,n,t,r)}$$

must be identically equal to

$$(54) \quad \Omega_{(m,n,t,r+1)}(i) j^{\eta(m,n,t,r+1)} k^{\lambda(m,n,t,r+1)} o^{\tau(m,n,t,r+1)}.$$

But by (49), (53) reduces to

$$\Omega_{(m,n,t,r)}(\chi) \mathfrak{A}_{(\eta,\lambda,\tau)}^{(8)}(i) j^{\mathcal{E}_{(\eta,\lambda,\tau)}^{(8)}} k^{\mathcal{F}_{(\eta,\lambda,\tau)}^{(8)}} o^{u_1\eta+u_2\lambda+u_3\tau},$$

where the arguments of η , λ , τ are (m, n, t, r) . Therefore we have the recursion formulas

$$(55) \quad \begin{aligned} \tau(m, n, t, r+1) &= u_1\eta + u_2\lambda + u_3\tau, \\ \lambda(m, n, t, r+1) &= \mathcal{F}_{(\eta,\lambda,\tau)}^{(8)}, \\ \eta(m, n, t, r+1) &= \mathcal{E}_{(\eta,\lambda,\tau)}^{(8)}, \\ \Omega_{(m,n,t,r+1)}(i) &= \Omega_{(m,n,t,r)}(\chi) \mathfrak{A}_{(\eta,\lambda,\tau)}^{(8)}, \end{aligned}$$

where η , λ , τ have arguments (m, n, t, r) . Since we may identify (71) for $r=0$ with (60), we have the initial values

$$(56) \quad \begin{aligned} \tau(m, n, t, 0) &= u_1m + u_2n + u_3t, \\ \lambda(m, n, t, 0) &= \mathcal{F}_{(m,n,t)}^{(8)}, \\ \eta(m, n, t, 0) &= \mathcal{E}_{(m,n,t)}^{(8)}, \\ \Omega_{(m,n,t,0)}(i) &= \mathfrak{A}_{(m,n,t)}^{(8)}(i). \end{aligned}$$

For these initial values, (55) and (56) serve as formulas to determine τ , λ , η , Ω completely. Now

$$(57) \quad \begin{aligned} j^{(h)}\gamma &= [\epsilon(\chi) J^y K^v O^u]^{(h-2)} \rho(i) j^c k^d o^l, \\ &= \epsilon(\chi^{h-1}) \mathfrak{A}^{(9)} j^{\mathcal{E}^{(9)}} k^{\mathcal{F}^{(9)}} o^{\tau(y,v,u,h-2)+l}. \end{aligned}$$

where every ϵ , y , v , u , has subscript 1. By 24₁

$$(58) \quad \begin{aligned} \gamma j &= \rho j^c k^d o^l j \\ &= \rho \mathcal{A}_{l1} \{\phi^d \theta^c\} B_{d,A(l,1)} \{\theta^c\} j^{c+\mathcal{F}^d A(l,1)} k^{d+L(l,1)} o^l. \end{aligned}$$

The condition that $j^{(h)}\gamma = \gamma j$ is therefore equivalent to

$$(59) \quad \begin{aligned} \epsilon_1(\chi^{h-1}) \mathfrak{A}^{(9)} j^{\mathcal{E}^{(9)}} k^{\mathcal{F}^{(9)}} o^{\tau(y_1,v_1,u_1,h-2)+l} \\ = \rho \mathcal{A}_{l1} \{\phi^d \theta^c\} B_{d,A(l,1)} \{\theta^c\} j^{c+\mathcal{F}^d A(l,1)} k^{d+L(l,1)} o^l. \end{aligned}$$

Condition that $k^{(h)}\gamma = \gamma k$. In (59), replace

$$\mathcal{H}^{(9)}, \mathcal{E}^{(9)}, \mathcal{F}^{(9)} \quad \text{by} \quad \mathcal{H}^{(10)}, \mathcal{E}^{(10)}, \mathcal{F}^{(10)},$$

$$\epsilon_1, y_1, v_1, u_1 \quad \text{by} \quad \epsilon_2, y_2, v_2, u_2,$$

$$A, A, L \quad \text{by} \quad \mathcal{D}, R, S.$$

The condition is

$$(60) \quad \epsilon_2(\chi^{h-1}) \mathcal{H}^{(10)} j \epsilon^{(10)} k \mathcal{F}^{(10)} o^{\tau(y_2, v_2, u_2, h-2)+l} \\ = \rho \mathcal{D}_{l1} \{ \phi^d \theta^c \} B_{d, R(l, 1)} \{ \theta^c \} j^{c+x^d R(l, 1)} k^{d+S(l, 1)} o^l.$$

Condition that $o^{(h)}\gamma = \gamma o$. In the left member of (59), replace

$$\mathcal{H}^{(9)}, \mathcal{E}^{(9)}, \mathcal{F}^{(9)} \quad \text{by} \quad \mathcal{H}^{(11)}, \mathcal{E}^{(11)}, \mathcal{F}^{(11)},$$

$$\epsilon_1, y_1, v_1, u_1 \quad \text{by} \quad \epsilon_3, y_3, v_3, u_3.$$

The condition is

$$(61) \quad \epsilon_3(\chi^{h-1}) \mathcal{H}^{(11)} j \epsilon^{(11)} k \mathcal{F}^{(11)} o^{\tau(y_3, v_3, u_3, h-2)+l} = \rho(i) j^c k^d o^{l+1}.$$

By [Theorem 1] with p replaced by h , we now have the following

THEOREM. Let an equation, $f(\xi) = 0$, be of degree $pqsh$, be irreducible in a field F , and have the roots (1) satisfying relations (2), (3), (4). Consider the algebra over F having the basis $i^m j^n k^t o^r E^r$ ($m \leq pqsh - 1$, $n \leq q - 1$, $t \leq p - 1$, $r \leq s - 1$, $r \leq h - 1$) with $f(i) = 0$, (5), (6), (7), and

$$E^h = \rho(i) j^c k^d o^l, \quad \begin{aligned} E^r P(i) &= P(\chi^r) E^r, \\ E^r j^t &= [j^{(r)}]^t E^r, \\ E^r k^t &= [k^{(r)}]^t E^r, \\ E^r o^t &= [o^{(r)}]^t E^r, \end{aligned}$$

where $j^{(r)}$, $k^{(r)}$, $o^{(r)}$ are equal to $\epsilon(\chi^{r-1}) (J^b K^r O^u)^{(r-2)}$ with subscripts 1, 2, 3, respectively on ϵ , y , v , u , where J , K , O are as defined in (11). The values of $j^{(r)}$, $k^{(r)}$, $o^{(r)}$ are found by (52), (55), (56) while their t -th powers may be found by (26), (29), (31) with altered parameters. This algebra is associative if and only if conditions [6], [14], [15], [18], [45], [49], [54], [61], [69], [72] and the conditions derived from (42), (43), (44), (45), (46), (47), (50), (59), (60), (61) by the application of formulas (39), (41) hold.

We note that these twenty associativity conditions are consistent, since they involve only products of the parameters; thus they are satisfied when all the parameters are equal to unity.

We do not include as conditions the facts that the constants c_1 , c_2 , c_3 ,

obtained in each case by the application of formulas (36), (37), (38), are integers, since these are conditions for the existence of the group.¹⁷

4. *Example. A group of order 32.*

Consider the group of order 32 defined abstractly as follows:¹⁸

$$\begin{aligned}\Theta^4 &= \Phi^2 = \Psi^2 = X^2 = 1, \\ \Theta\Phi &= \Phi\Theta, & \Theta X &= X\Phi\Theta^3, \\ \Theta\Psi &= \Psi\Phi\Theta, & \Phi X &= X\Phi, \\ \Phi\Psi &= \Psi\Phi, & \Psi X &= X\Psi.\end{aligned}$$

This group has an abelian subgroup of order 16, type (1, 1, 1, 1) with generators Θ, Φ, Ψ, X .

Let an equation $f(\xi) = 0$ of degree 32 be irreducible in a field F and have the roots

$$\chi^{a_1}\{\psi^{a_2}\phi^{a_3}\theta^{a_4}\} \quad (a_1 < 2, a_2 < 2, a_3 < 2, a_4 < 4)$$

satisfying the following relations:

$$\begin{aligned}\theta^4(i) &= \phi^2(i) = \psi^2(i) = \chi^2(i) = i, \\ \theta\{\phi\} &= \phi\{\theta\}, & \theta\{\chi\} &= \chi\{\phi\theta^3\}, \\ \theta\{\psi\} &= \psi\{\phi\theta\}, & \phi\{\chi\} &= \chi\{\phi\}, \\ \phi\{\psi\} &= \psi\{\phi\}, & \psi\{\chi\} &= \chi\{\psi\}.\end{aligned}$$

Then the group constants of the general discussion have the values

$$\begin{aligned}q &= 4 & x &= 1, \\ p &= 2 & z_1 &= 1, w_1 = 1, \\ s &= 2 & z_2 &= 1, w_2 = 0, \\ h &= 2 & u_1 &= 0, v_1 = 1, y_1 = 3, \\ e &= a = b = c = d = l = 0, & u_2 &= 0, v_2 = 1, y_2 = 0, \\ & & u_3 &= 1, v_3 = 0, y_3 = 0.\end{aligned}$$

The constants c_1 to c_{14} of Dickson's paper are all zero except

$$c_3 = 2, \quad c_4 = 1, \quad c_{11} = 1.$$

The table (32) becomes

17. Dickson, "Construction of Division Algebras," Theorem 6.

18. This group was suggested by Dr. J. K. Senior as one which probably cannot be represented with fewer than four generators.

Superscript	ϵ_a	y_a	v_a	u_a	ϵ_b	y_b	v_b	u_b	$\mathcal{F}$	$\mathcal{E}$	$\mathcal{A}$
(1)	ϵ_1	3	1	0	ϵ_2	0	1	0	2	3	$\epsilon_1 \epsilon_2 \{\phi \theta^3\}$
(2)	ϵ_2	0	1	0	ϵ_3	0	0	1	1	0	$\epsilon_2 \epsilon_3 \{\phi\}$
(3)	ϵ_1	3	1	0	$\epsilon_2 \epsilon_3 \{\phi\}$	0	1	1	2	3	$\epsilon_1 \epsilon_2 \epsilon_3 \{\theta^3\}$
(4)	ϵ_2	0	1	0	ϵ_3	0	0	1	1	0	$\epsilon_2 \epsilon_3 \{\phi\}$
(5)	1	0	0	0	$\epsilon_2 \epsilon_3 \{\phi\}$	0	1	1	1	0	$\epsilon_2 \epsilon_3 \{\phi\}$
(6)	1	0	0	0	1	0	0	0	0	0	1
(7)	00	1	0	0	0	1	0	0	0	0	1
	10	ϵ_2	0	1	0	1	0	0	0	1	ϵ_2
	01	1	0	0	0	ϵ_3	0	0	1	0	ϵ_3
(8)	000	1	0	0	0	1	0	0	0	0	1
	310	Δ	9	3	0	ϵ_2	0	1	0	4	$\Delta \epsilon_2 \{\phi \theta\}$
	010	1	0	0	0	ϵ_2	0	1	0	1	ϵ_2
	001	1	0	0	0	ϵ_3	0	0	1	0	ϵ_3
(9)	$\Delta \epsilon_2 \{\phi \theta\}$	9	4	0	ρ	0	0	0	4	9	$\Delta \epsilon_2 \{\phi \theta\} \rho \{\theta\}$
(10)	ϵ_2	0	1	0	ρ	0	0	0	1	0	$\epsilon_2 \rho \{\phi\}$
(11)	ϵ_3	0	0	1	ρ	0	0	0	0	0	$\epsilon_3 \rho \{\psi\}$

where $\Delta \equiv \epsilon_1 \epsilon_1 \{\phi \theta^3\} [\alpha(i) \alpha(\theta^3)]^2 \alpha\{\theta\} \alpha\{\phi\} \alpha\{\theta^2\} \alpha\{\phi \theta^2\} \alpha\{\phi \theta^3\} \epsilon_1 \{\theta^2\}$.

After evaluating the exponents, we find the conditions in this paper for the present example. They are

$$(42) \quad \mathcal{H}_{21}(i) j^3 k^2 o^0 = \alpha(\chi) \mathcal{H}^{(1)}(i) j^3 k^2 o^0,$$

$$(43) \quad \mathcal{H}_{31}(i) j^3 k^1 o^1 = \epsilon_1(\chi) \mathcal{H}^{(3)}(i) j^3 k^2 o^1,$$

$$(44) \quad \mathcal{H}_{32}(i) j^0 k^1 o^1 = \epsilon_2(\chi) \mathcal{H}^{(5)}(i) j^0 k^1 o^1,$$

$$(45) \quad \mathcal{G}_1^{(4)}(i) j^{12} k^4 o^0 = g(\chi),$$

$$(46) \quad \mathcal{G}_2^{(2)}(i) j^0 k^2 o^0 = \beta(\chi) \mathcal{G}_1^{(0)}(i) j^0 k^0 o^0,$$

$$(47) \quad \mathcal{G}_3^{(2)}(i) j^0 k^0 o^2 = \sigma(\chi) \mathcal{H}^{(6)}(i) j^0 k^0 o^0,$$

$$(50) \quad \rho(i) j^0 k^0 o^0 = \rho(\chi) \mathcal{H}^{(8)}(i) j^0 k^0 o^0,$$

$$(59) \quad \epsilon_1(\chi) \mathcal{H}^{(9)} j^0 k^4 o^0 = \rho \mathcal{A}_{01} B_{0,1} j^1 k^0 o^0,$$

$$(60) \quad \epsilon_2(\chi) \mathcal{H}^{(10)} j^0 k^1 o^0 = \rho \mathcal{D}_{01} B_{0,0} j^0 k^1 o^0,$$

$$(61) \quad \epsilon_3(\chi) \mathcal{H}^{(11)} j^0 k^0 o^1 = \rho(i) j^0 k^0 o^1.$$

We may now determine completely the associativity conditions for our algebra.

Associativity conditions.

$$[6] \quad g(i) = g(\theta).$$

$$[14] \quad \alpha(i) \alpha(\theta) \alpha(\theta^2) \alpha(\theta^3) g(i) = g(\phi).$$

- [15] $\beta(\phi) = \beta(i).$
- [18] $\alpha(\phi)\alpha(i)\beta(\theta) = \beta(i).$
- [45] $\delta_1\delta_1(\phi\theta)\delta_1(\theta^2)\delta_1(\phi\theta^3)\alpha(\theta)\alpha(\theta^2)[\alpha(\theta^3)]^2\alpha(\phi\theta^2)\alpha(\phi\theta^3)[\beta(i)]^2g(i) = g(\psi).$
- [49] $\alpha(\psi)\delta_1\delta_2(\phi\theta) = \delta_2\delta_1(\phi)\alpha(i).$
- [54] $\delta_2\delta_2(\phi)\beta(i) = \beta(\psi).$
- [61] $\sigma(\psi) = \sigma(i).$
- [69] $\delta_1\delta_1(\psi)\delta_2(\phi\theta)\sigma(\theta)\beta(\theta) = \sigma(i).$
- [72] $\delta_2\delta_2(\psi)\sigma(\phi) = \sigma(i).$
- (39) $\epsilon_2\epsilon_1(\phi)\alpha\alpha(\theta)\alpha(\theta^2) = \alpha(\chi)\epsilon_1\epsilon_2(\phi\theta^3).$
- (43) $\epsilon_3\epsilon_1(\psi)\delta_1\delta_1(\phi\theta)\delta_1(\theta^2)\delta_2(\phi\theta^3)\alpha(\theta)\alpha(\phi\theta^2)\alpha(\theta^2)\beta(\theta^3) = \epsilon_1(\chi)\epsilon_1\epsilon_2\epsilon_3(\theta^3).$
- (45) $\epsilon_3\epsilon_2(\psi)\delta_2 = \epsilon_2(\chi)\epsilon_2\epsilon_3(\phi).$
- (50) $\epsilon_1\epsilon_1(\theta^2)\epsilon_1(\phi\theta)\epsilon_1(\phi\theta^3)\alpha\alpha(\phi)\alpha(\phi\theta)$
 $\times [\alpha(\theta)]^3[\alpha(\theta^2)]^4[\alpha(\theta^3)]^4[\alpha(\phi\theta^3)]^2[\alpha(\phi\theta^2)]^2\beta^2g^3 = g(\chi).$
- (54) $\epsilon_2\epsilon_2(\phi)\beta(i) = \beta(\chi).$
- (58) $\epsilon_3\epsilon_3(\psi)\sigma(i) = \sigma(\chi).$
- (65) $\rho(i) = \rho(\chi).$
- (76) $\epsilon_1(\chi)\epsilon_1\epsilon_1(\phi\theta^3)\epsilon_1(\theta^2)\epsilon_2(\phi\theta)\alpha(\theta)\alpha(\phi\theta^2)$
 $\times \alpha(\phi\theta^3)\alpha(\phi)\alpha(\theta^2)[\alpha(i)\alpha(\theta^3)\beta(\theta)]^2\rho(\theta)g^2 = \rho(i).$
- (77) $\epsilon_2\epsilon_2(\chi)\rho(\phi) = \rho(i).$
- (81) $\epsilon_3\epsilon_3(\chi)\rho(\psi) = \rho(i).$

If we take $g = \beta = \sigma = \rho = \alpha = \delta_1 = \delta_2 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, the algebra over F determined by our equation, $f(\xi) = 0$, has the basis

$$i^m j^n k^t o^f E^r \quad (m \leq 31, n \leq 3, t \leq 1, f \leq 1, r \leq 1),$$

with $f(i) = 0$; and the multiplication table

$$\begin{aligned} j^4 &= 1, & jP(i) &= P(\theta)j, & j^2P(i) &= P(\theta^2)j^2, & j^3P(i) &= P(\theta^3)j^3, \\ k^2 &= 1, & kP(i) &= P(\phi)k, \\ & & kj &= jk, & kj^2 &= j^2k, & kj^3 &= j^3k, \\ o^2 &= 1, & oP(i) &= P(\psi)o, \\ & & oj &= jko, & oj^2 &= j^2o, & oj^3 &= j^3ko, \\ & & ok &= ko, \\ E^2 &= 1, & EP(i) &= P(\chi)E, \\ & & Ej &= j^3kE, & Ej^2 &= j^2E, & Ej^3 &= jkE, \\ & & Ek &= kE, \\ & & Eo &= oE. \end{aligned}$$

This algebra is associative.

VITA.

Mina Spiegel Rees was born in Cleveland, Ohio, on August 2, 1902. She attended the public schools of New York City, and received the degree of Bachelor of Arts from Hunter College of the City of New York in 1923. In 1925, she received the degree of Master of Arts from Columbia University. She was appointed instructor in Mathematics at Hunter College in 1926. From this position she received a leave of absence in 1929 to study at the University of Chicago, where she has been in continuous residence since that time.

At the University of Chicago she has studied under Professors Barnard, Bliss, Dickson, Graves, Lane, Logsdon, Lunn, MacMillan and Slaught. To all of these she acknowledges her indebtedness, but particularly to Professor Dickson, under whose direction this thesis was written, and for whose interest and encouragement she is sincerely grateful.

